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BRST quantization of gauge theories like SL(2,R) on inner product spaces

Robert Marnelius[1] and Ulrich Quaade[2]

Institute of Theoretical Physics
Chalmers University of Technology
S-412 96 Göteborg, Sweden

Abstract

Some general formulas are derived for the solutions of a BRST quantization on inner product spaces of finite dimensional bosonic gauge theories invariant under arbitrary Lie groups. A detailed analysis is then performed of SL(2,R) invariant models and some possible geometries of the Lagrange multipliers are derived together with explicit results for a class of SL(2,R) models. Gauge models invariant under a nonunimodular gauge group are also studied in some detail.

[1]tferm@fy.chalmers.se
[2]Permanent address: Odense Univ., Campusv 55, DK-5230 Odense M. E-mail: ujq@dou.dk

1 Introduction.

In ref.[1] it was shown that a BRST quantization of a bosonic gauge theory with an arbitrary Lie group invariance may be explicitly solved on an inner product space for finite degrees of freedom. The solutions were shown to have the form

$$|ph\rangle = e^{[\rho,Q]_+}|\Phi\rangle \tag{1.1}$$

where Q is the BRST charge operator and ρ a fermionic gauge fixing operator. $|\Phi\rangle$ is a state which does not depend on ghosts or Lagrange multipliers. Eq.(1.1) is a formal solution which actually may be extended to gauge theories of arbitrary ranks [2]. In order to find out whether or not it exists one must prescribe exactly the quantization properties of the involved variables. In [3] some quantization rules were proposed which were shown to be necessary in order for states of the form (1.1) to have finite norms. These rules determine the quantum properties of the original inner product space with indefinite metric states.

In this paper we derive some general formulas for the solutions (1.1) which are to be used in conjunction with the quantization rules of [3]. In particular we perform a detailed analysis of general gauge models whose gauge group is SL(2,R) and determine some possible geometries of the Lagrange multipliers. The results for an explicit class of models are also derived. Finally we treat gauge models invariant under a simple nonunimodular gauge group.

2 General properties.

Consider a gauge theory in which the gauge generators $\psi_a,\ a = 1, \ldots, m$ are bosonic and satisfy the Lie algebra

$$[\psi_a, \psi_b]_- = iU_{ab}^{\ \ c}\psi_c \tag{2.1}$$

where $U_{ab}^{\ \ c}$ are the structure constants. The BFV form of the BRST charge operator is then [4]

$$Q = \psi_a\eta^a - \frac{1}{2}iU_{bc}^{\ \ a}\mathcal{P}_a\eta^b\eta^c - \frac{1}{2}iU_{ab}^{\ \ b}\eta^a + \bar{\mathcal{P}}_a\pi^a \tag{2.2}$$

where η^a, $\bar{\eta}_a$ are Faddeev-Popov ghosts and antighosts respectively and $\mathcal{P}_a$, $\bar{\mathcal{P}}^a$ their conjugate momenta. π_a are conjugate momenta to the Lagrange multipliers v^a. They satisfy the algebra (the nonzero part)

$$[\eta^a, \mathcal{P}_b]_+ = [\bar{\eta}^a, \bar{\mathcal{P}}_b]_+ = \delta^a_b, \quad [v^a, \pi_b]_- = i\delta^a_b \tag{2.3}$$

We assume that ψ_a as well as all ghosts and Lagrange multipliers are hermitian which is no restriction. This implies that Q in (2.2) is hermitian. In [1] it was shown that the general Q given in (2.2) may always be decomposed as

$$Q = \delta + \delta^\dagger \tag{2.4}$$

where

$$\delta^2 = \delta^{\dagger 2} = [\delta, \delta^\dagger]_+ = 0 \tag{2.5}$$

Based on the results of [5] it was also shown that the BRST condition $Q|ph\rangle = 0$, which projects out the physical states $|ph\rangle$, may be solved by a bigrading which requires

$$\delta|ph\rangle = 0, \quad \delta^{\dagger}|ph\rangle = 0 \tag{2.6}$$

The decomposition (2.4) is not unique. However, for the class of decompositions found in [1, 6] the solutions of (2.6) could be written as

$$|ph\rangle_{\alpha} = e^{\alpha[\rho,Q]_{+}}|\Phi\rangle \tag{2.7}$$

where α is a real constant different from zero, and ρ a hermitian fermionic gauge fixing operator whose possible forms were derived. For the decompositions found in [1] it is of the form

$$\rho = \mathcal{P}_a v^a \tag{2.8}$$

and where $|\Phi\rangle$ satisfies the conditions

$$\eta^a|\Phi\rangle = \bar{\eta}_a|\Phi\rangle = \pi_a|\Phi\rangle = 0 \tag{2.9}$$

which simply means that $|\Phi\rangle$ has no ghost dependence and no dependence on the Lagrange multipliers v^a. Thus, $|\Phi\rangle$ only depends on the matter variables. Since (2.9) is always trivially solved, the expression (2.7) represents general solutions for any gauge theory with finite number of degrees of freedom and in which the gauge group is a Lie group. (In [2] the solution (2.7) is extended to any gauge group.) For the decompositions (2.4) found in [6] there are also formal solutions of the form (2.7), however, in this case ρ involves gauge fixing conditions to ψ_a and $|\Phi\rangle$ must be a solution of a Dirac quantization ($\sim \psi_a|\Phi\rangle = 0$). Although these are also possible solutions we concentrate on the solutions (2.7) where ρ and $|\Phi\rangle$ satisfy (2.8) and (2.9). They are of a more geometrical nature since ρ is simple and $|\Phi\rangle$ trivial.

The inner products of the solutions (2.7) have the form

$${}_{\alpha}\langle ph|ph\rangle_{\alpha} = \langle\Phi|e^{2\alpha[\rho,Q]_{+}}|\Phi\rangle \tag{2.10}$$

From (2.9) it follows that this expression can only be well defined if $\alpha \neq 0$ ($|\Phi\rangle$ does not belong to an inner product space). However, one may easily prove that (2.10) is independent of the value of α as long as it is positive or negative [1, 2]. In fact, we have for the gauge fixing (2.8) [2]

$$|ph\rangle_{\alpha} = U(\beta)|ph\rangle_{\pm}, \quad |ph\rangle_{\pm} \equiv |ph\rangle_{\pm 1} \tag{2.11}$$

where $\alpha = \pm e^{\beta}$ and where $U(\beta)$ is the unitary operator

$$U(\beta) \equiv e^{-i\beta[v^a\bar{\eta}_a,Q]_{+}} \tag{2.12}$$

Eq.(2.11) means that (2.10) may only depend on the sign of α. As in [3] it is natural to impose the condition that (2.10) is independent of the sign of α. However, this is a nontrivial condition. Notice *e.g.* that there cannot be solutions $|ph\rangle_{\alpha}$ for both positive and negative α's at the same time if we have performed a projection from an original inner product space since ${}_{-\alpha}\langle ph|ph\rangle_{\alpha} = \langle\Phi|\Phi\rangle$ is undefined. In fact, as was clarified in [3], the original state spaces yielding the solutions with positive and negative α have different

bases. The condition that solutions with opposite signs of α should yield equivalent results requires a rather precise general condition on the ranges of the Lagrange multipliers (see below and [3]).

Consider now the inner product (2.10) for the gauge fixing (2.8) with $\alpha = \frac{1}{2}$. We have

$$[\rho, Q]_+ = \psi_a v^a + \psi^{gh}_a v^a + i\mathcal{P}_a\bar{\mathcal{P}}^a, \quad \psi^{gh}_a \equiv \frac{1}{2} i U_{ab}{}^c(\mathcal{P}_c\eta^b - \eta^b\mathcal{P}_c) \tag{2.13}$$

In appendix A the following two equivalent expressions are derived

$$_+\langle ph|ph\rangle_+ = \langle\Phi|e^{[\rho,Q]_+}|\Phi\rangle = \begin{cases} \langle\Phi|e^{\psi'_a v^a} e^{i(L^{-1})^a{}_b(iv)\mathcal{P}_a\bar{\mathcal{P}}^b}|\Phi\rangle \\ \langle\Phi|e^{i(L^{\dagger-1})^a{}_b(iv)\mathcal{P}_a\bar{\mathcal{P}}^b} e^{\psi''_a v^a}|\Phi\rangle \end{cases} \tag{2.14}$$

where $L^a{}_b(iv)$ is the left invariant vielbein on the group manifold with imaginary coordinates iv^a ($L^{\dagger a}{}_b(iv) = L^a{}_b(-iv)$) (see also [1]), and where

$$\psi'_a \equiv \psi_a - \frac{1}{2} i U_{ab}{}^b, \quad \psi''_a \equiv \psi_a + \frac{1}{2} i U_{ab}{}^b = (\psi'_a)^\dagger \tag{2.15}$$

Notice that

$$e^{i(L^{\dagger-1})^a{}_b(iv)\mathcal{P}_a\bar{\mathcal{P}}^b} e^{\psi''_a v^a} = (e^{\psi'_a v^a} e^{i(L^{-1})^a{}_b(iv)\mathcal{P}_a\bar{\mathcal{P}}^b})^\dagger \tag{2.16}$$

For $\alpha = -\frac{1}{2}$ in (2.10) we find in appendix A the expressions

$$_-\langle ph|ph\rangle_- = \langle\Phi|e^{-[\rho,Q]_+}|\Phi\rangle = \begin{cases} \langle\Phi|e^{-\psi'_a v^a} e^{-i(L^{\dagger-1})^a{}_b(iv)\mathcal{P}_a\bar{\mathcal{P}}^b}|\Phi\rangle \\ \langle\Phi|e^{-i(L^{-1})^a{}_b(iv)\mathcal{P}_a\bar{\mathcal{P}}^b} e^{-\psi''_a v^a}|\Phi\rangle \end{cases} \tag{2.17}$$

where

$$e^{-i(L^{-1})^a{}_b(iv)\mathcal{P}_a\bar{\mathcal{P}}^b} e^{-\psi''_a v^a} = (e^{-\psi'_a v^a} e^{-i(L^{\dagger-1})^a{}_b(iv)\mathcal{P}_a\bar{\mathcal{P}}^b})^\dagger \tag{2.18}$$

Now we have

$$e^{iM^a{}_b\mathcal{P}_a\bar{\mathcal{P}}^b} = (\det M^a{}_b)(i\mathcal{P}_a\bar{\mathcal{P}}^a)^m + \text{lower powers of } \mathcal{P}_a \text{ and } \bar{\mathcal{P}}^a \tag{2.19}$$

and

$$|\Phi\rangle = |\phi\rangle|0\rangle_\pi|0\rangle_{\eta\bar{\eta}} \tag{2.20}$$

where $|0\rangle_\pi$ is a Lagrange multiplier vacuum ($\pi_a|0\rangle = 0$), $|0\rangle_{\eta\bar{\eta}}$ a ghost vacuum ($\eta^a|0\rangle_{\eta\bar{\eta}} = \bar{\eta}_a|0\rangle_{\eta\bar{\eta}} = 0$) and $|\phi\rangle$ an arbitrary matter state which is spanned by the matter variables involved in *e.g.* ψ_a. (The physical part of $|\phi\rangle$ should belong to an inner product space.) The ghost vacuum may be normalized as follows [7]

$$_{\eta\bar{\eta}}\langle 0|(i\mathcal{P}_a\bar{\mathcal{P}}^a)^m|0\rangle_{\eta\bar{\eta}} = 1 \tag{2.21}$$

Making use of (2.19) and (2.20) in (2.14) and (2.17) we find

$$_+\langle ph|ph\rangle_+ = \begin{cases} \langle\phi|\,_\pi\langle 0|e^{\psi'_a v^a} \det (L^{-1})^a{}_b(iv)|0\rangle_\pi|\phi\rangle \\ \langle\phi|\,_\pi\langle 0|\det (L^{\dagger-1})^a{}_b(iv)e^{\psi''_a v^a}|0\rangle_\pi|\phi\rangle \end{cases} \tag{2.22}$$

and

$$
{}_-\langle ph|ph\rangle_- = \left\{ \begin{array}{c} (-1)^m \langle\phi|\,{}_\pi\langle 0|e^{-\psi'_a v^a}\det{(L^{\dagger -1})^a}_b(iv)|0\rangle_\pi|\phi\rangle \\ (-1)^m \langle\phi|\,{}_\pi\langle 0|\det{(L^{-1})^a}_b(iv)e^{-\psi''_a v^a}|0\rangle_\pi|\phi\rangle \end{array} \right. \tag{2.23}
$$

The Lagrange multipliers v^a are hermitian operators. They have therefore either real or imaginary eigenvalues depending on whether or not they span a positive or an indefinite metric state space. In the special case when all v's span positive metric states, the eigenstates may be normalized as follows

$$
\langle v|v'\rangle = \delta^m(v-v'), \quad \int d^m v|v\rangle\langle v| = \mathbf{1} \tag{2.24}
$$

Choosing the normalization of $|0\rangle_\pi$ such that

$$
{}_\pi\langle 0|v\rangle = \langle v|0\rangle_\pi = 1 \tag{2.25}
$$

we find for (2.22) and (2.23)

$$
{}_+\langle ph|ph\rangle_+ = \left\{ \begin{array}{c} \int d^m v \det{(L^{-1})^a}_b(iv)\langle\phi|e^{\psi'_a v^a}|\phi\rangle \\ \int d^m v \det{(L^{\dagger -1})^a}_b(iv)\langle\phi|e^{\psi''_a v^a}|\phi\rangle \end{array} \right. \tag{2.26}
$$

$$
{}_-\langle ph|ph\rangle_- = \left\{ \begin{array}{c} (-1)^m \int d^m v \det{(L^{-1})^a}_b(-iv)\langle\phi|e^{-\psi'_a v^a}|\phi\rangle \\ (-1)^m \int d^m v \det{(L^{\dagger -1})^a}_b(-iv)\langle\phi|e^{-\psi''_a v^a}|\phi\rangle \end{array} \right. \tag{2.27}
$$

In this case it is clear that convergence of these expressions will in general severely restrict the ranges of v^a.

In the other extreme case when all Lagrange multipliers span indefinite metric state spaces, v^a has imaginary eigenvalues iu^a. Such eigenstates satisfy the properties [8, 9]

$$
\begin{aligned}
& v^a|iu\rangle = iu^a|iu\rangle, \quad \langle -iu| = (|iu\rangle)^\dagger \\
& \langle iu'|iu\rangle = \delta^m(u'-u), \quad \int d^m u|-iu\rangle\langle -iu| = \int d^m u|iu\rangle\langle iu| = \mathbf{1}
\end{aligned} \tag{2.28}
$$

Notice that these relations requires u^a to have symmetric ranges around zero. Inserting the last completeness relations in (2.22) and (2.23) we obtain again using a normalization like (2.25):

$$
\begin{aligned}
& {}_+\langle ph|ph\rangle_+ = \left\{ \begin{array}{c} \int d^m u \det{(R^{-1})^a}_b(u)\langle\phi|e^{i\psi'_a u^a}|\phi\rangle \\ \int d^m u \det{(L^{-1})^a}_b(u)\langle\phi|e^{i\psi''_a u^a}|\phi\rangle \end{array} \right. \\
& {}_-\langle ph|ph\rangle_- = (-1)^m\, {}_+\langle ph|ph\rangle_+
\end{aligned} \tag{2.29}
$$

where ${R^a}_b(u) = {L^a}_b(-u)$ and where the last relation follows since the ranges of u^a are symmetric around zero. These are the natural gauge theoretic expressions: $\det{(R^{-1})^a}_b(u)$ and $\det{(L^{-1})^a}_b(u)$ are the right and left-invariant group measures. u^a canonical coordinates (which also are symmetric around zero) and $e^{i\psi'_a u^a}$ a finite group element belonging to the identity component of the group. Equality for positive and negative α's requires that the normalization of the two matter vacua are related by a factor $(-1)^m$ (see section 4 of [3]).

For unimodular gauge groups we have

$$U_{ab}{}^{b} = 0 \tag{2.30}$$

which implies

$$\psi''_a = \psi'_a = \psi_a \tag{2.31}$$

and

$$\det{(L^{\dagger -1})^a}_b(iv) \equiv \det{(L^{-1})^a}_b(-iv) = \det{(L^{-1})^a}_b(iv) \tag{2.32}$$

In this case the two equivalent expressions on the right-hand side of (2.26), (2.27) and (2.29) coincide. (In particular we have $\det R^{-1} = \det L^{-1}$ in (2.29).)

Exactly how one should choose to quantize the Lagrange multipliers v^a is not given beforehand. In fact it is not given how to quantize any of the involved variables beforehand. However, in [3] the following necessary condition for the possible existence of the solution (1.1) was given (from the requirement that (1.1) is an inner product state):

The Lagrange multipliers v^a must be quantized with opposite metric states to the unphysical variables which the gauge generator ψ_a eliminates. (2.33)

For trivial models it was shown in [3] that all such possible choices lead to the same result for a fixed choice of the quantization of the gauge invariant physical matter variables. This is not the case for more involved models where usually only *particular choices make the formal solutions exist*. In [3] some simple, essentially abelian examples were given. In the following sections we shall treat $SU(2)$ and $SL(2,R)$ gauge theories in details.

In order for the physical states to have positive norms one must quantize the remaining matter variables with positive metric states. This is a necessary condition. One may notice that this together with (2.33) require $|\phi\rangle$ in (2.29) to be expandable in terms of positive metric states while this is not the case in (2.26)-(2.27). Maybe this provide for the possibility to give a general proof that the physical states (2.29) have positive norms whenever they exist (which should be the case at least for compact gauge groups).

We end this section with a technical remark. In order to calculate (2.26) and (2.29) explicitly one has to factorize the exponents $e^{\psi_a v^a}$ and $e^{-\psi_a v^a}$ in (2.22) and (2.23), *i.e.* one has to determine new parameters w_i in relations like

$$e^{\psi_a v^a} = e^{w_n^\dagger A_n^\dagger} \cdots e^{w_2^\dagger A_2^\dagger} e^{w_1 A_1} e^{w_2 A_2} \cdots e^{w_n A_n} \tag{2.34}$$

where A_i are linear expressions in ψ_a ($A_i \neq A_{i+1}, A_{i-1}$), and w_i expressions in terms of the Lagrange multipliers v^a. A_1 and w_1 must be hermitian since $e^{\psi_a v^a}$ is hermitian. Notice that $e^{-\psi_a v^a}$ is the inverse operator to $e^{\psi_a v^a}$. We have therefore

$$e^{-\psi_a v^a} = e^{-w_n A_n} \cdots e^{-w_2 A_2} e^{-w_1 A_1} e^{-w_2^\dagger A_2^\dagger} \cdots e^{-w_n^\dagger A_n^\dagger} \tag{2.35}$$

Hence, the transformation $v^a \rightarrow -v^a$ is accompanied by $w_1 \rightarrow -w_1$ and $w_i \rightarrow -w_i^\dagger$ for $i \geq 2$. Locally we shall always require that v^a is determined by w_i. This requires in turn that $n \geq (m+1)/2$ where m is the number of Lagrange multipliers v^a. Now, after a choice of quantization of the original state space one will find that w_i and v^a are not

in a one-to-one correspondence if the ranges are not restricted. These restrictions will depend on the gauge group, the choice of quantization and on the choice of factorization (the choice of A_i in (2.34)). Our results suggest that the gauge group and the choice of quantization essentially determines the appropriate factorization. This is natural since the choice of factorization is only a technical point which cannot determine the quantum theory. A given gauge model determines, of course, the factorization and the possible choices of quantization. This procedure will be treated in detail for $SU(2)$ and $SL(2,R)$ gauge theories in the following sections. Also a nonunimodular group will be treated.

3 SU(2) gauge theories

In this section we give some general properties of an $SU(2)$ gauge theory. Consider the case when the gauge generators ψ_a satisfy the Lie algebra (since the group metric is Euclidean we use only lower indices)

$$[\psi_a, \psi_b]_- = i\varepsilon_{abc}\psi_c \tag{3.1}$$

where ε_{abc} is the three dimensional antisymmetric symbol ($\varepsilon_{123} = 1$). The BRST charge operator is then

$$Q = \psi_a\eta_a - \frac{1}{2}i\varepsilon_{abc}\mathcal{P}_a\eta_b\eta_c + \bar{\mathcal{P}}_a\pi_a \tag{3.2}$$

With the gauge fixing $\rho = \mathcal{P}_a v_a$ we find

$$[\rho, Q]_+ = \psi_a v_a + \frac{1}{2}i\varepsilon_{abc}(\mathcal{P}_c\eta_b - \eta_b\mathcal{P}_c)v_a + i\mathcal{P}_a\bar{\mathcal{P}}_a \tag{3.3}$$

For compact semisimple gauge groups, which are unimodular, the norms of the physical states are determined by the expressions

$$\begin{aligned}
{}_+\langle ph|ph\rangle_+ &= \langle\Phi|e^{[\rho,Q]_+}|\Phi\rangle = \langle\phi|\,{}_\pi\langle 0|e^{\psi_a v_a}\det{(L^{-1})}_{ab}(iv)|0\rangle_\pi|\phi\rangle\\
{}_-\langle ph|ph\rangle_- &= \langle\Phi|e^{-[\rho,Q]_+}|\Phi\rangle = (-1)^m\langle\phi|\,{}_\pi\langle 0|e^{-\psi_a v_a}\det{(L^{-1})}_{ab}(iv)|0\rangle_\pi|\phi\rangle
\end{aligned} \tag{3.4}$$

from (2.22)-(2.23) and (2.30)-(2.32). For $SU(2)$ the left invariant vielbein $(L^{-1})_{ab}$ has the explicit form ($v = \sqrt{v_1^2 + v_2^2 + v_3^2}$)

$$(L^{-1})_{ab}(iv) = \delta_{ab}\frac{\sinh v}{v} - \frac{v_a v_b}{v^2}(\frac{\sinh v}{v} - 1) - i\varepsilon_{abc}v_c\frac{\cosh v - 1}{v^2} \tag{3.5}$$

from which we find the measure operator

$$\det(L^{-1})_{ab}(iv) = \frac{2(\cosh v - 1)}{v^2} \tag{3.6}$$

which satisfies the unimodularity properties (2.30) and (2.32). It is natural to restrict the possible spectra of v_a to those which make the measure operator (3.6) real. Here this requires that v_a for each value of a should have either real or imaginary eigenvalues. For real spectra of all three Lagrange multipliers v_a we find

$$\begin{aligned}
{}_+\langle ph|ph\rangle_+ &= \int d^3v\frac{2(\cosh v - 1)}{v^2}\langle\phi|e^{\psi_a v_a}|\phi\rangle\\
{}_-\langle ph|ph\rangle_- &= -\int d^3v\frac{2(\cosh v - 1)}{v^2}\langle\phi|e^{-\psi_a v_a}|\phi\rangle
\end{aligned} \tag{3.7}$$

and for imaginary spectra ($v_a \to i u_a$, $u=\sqrt{u_1^2+u_2^2+u_3^2}$)

$$
\begin{aligned}
{}_+\langle ph|ph\rangle_+ &= \int d^3u \frac{2(1-\cos u)}{u^2}\langle\phi|e^{i\psi_a u_a}|\phi\rangle \\
{}_-\langle ph|ph\rangle_- &= -{}_+\langle ph|ph\rangle_+
\end{aligned}
\tag{3.8}
$$

These are the two natural choices for an SU(2) gauge theory since we have to restrict the spectra of v^a so that $v=\sqrt{v_1^2+v_2^2+v_3^2}$ is either real or imaginary. (The transition between these two possibilities is double valued, $v\to \pm\, iu$.) Furthermore, since the last measure is zero for $u=2\pi n,\ n=\pm1,\pm2,...$ it seems in this case also natural to restrict the ranges of u_a within the limits $0\leq u<2\pi$.

In order to calculate the matrix elements $\langle\phi|e^{\pm\psi_a v_a}|\phi\rangle$ in (3.7)-(3.8) one needs in general to factorize the exponential operator (3.4) like (2.34). We may *e.g.* choose the Euler angle like decomposition

$$
e^{\psi_a v_a} = e^{\alpha\psi_3}e^{\beta\psi_2}e^{\gamma\psi_3}
\tag{3.9}
$$

where α,β and γ are expressions in terms of v_a. The right-hand side is manifestly hermitian if

$$
\beta^\dagger=\beta, \quad \gamma=\alpha^\dagger
\tag{3.10}
$$

which we require. Eq.(3.9) is easily solved algebraically. We find

$$
\begin{aligned}
\frac{v^1}{v}\sinh\frac{v}{2} &= -i\sinh\frac{\beta}{2}\sinh\frac{\alpha-\gamma}{2}, \quad \frac{v^2}{v}\sinh\frac{v}{2}=\sinh\frac{\beta}{2}\cosh\frac{\alpha-\gamma}{2} \\
\frac{v^3}{v}\sinh\frac{v}{2} &= \cosh\frac{\beta}{2}\sinh\frac{\alpha+\gamma}{2}, \quad \cosh\frac{v}{2}=\cosh\frac{\beta}{2}\cosh\frac{\alpha+\gamma}{2}.
\end{aligned}
\tag{3.11}
$$

All these relations are consistent with hermitian v_a and (3.10).

For the case (3.7) we find then with $\alpha=\xi+i\zeta$ and $\xi,\ \zeta$ real

$$
\begin{aligned}
{}_+\langle ph|ph\rangle_+ &= \frac{1}{2}\int d\xi d\beta d\zeta \sinh\beta\langle\phi|e^{(\xi+i\zeta)\psi_3}e^{\beta\psi_2}e^{(\xi-i\zeta)\psi_3}|\phi\rangle \\
{}_-\langle ph|ph\rangle_- &= -{}_+\langle ph|ph\rangle_+
\end{aligned}
\tag{3.12}
$$

where $\beta\geq0$ and $-\infty<\xi,\zeta<\infty$ for infinite ranges of the Lagrange multipliers v_a.

For the case (3.8) we find on the other hand the standard Euler angle decomposition (α,β,γ real)

$$
\begin{aligned}
{}_+\langle ph|ph\rangle_+ &= \int d\alpha d\beta d\gamma \sin\beta\langle\phi|e^{i\alpha\psi_3}e^{i\beta\psi_2}e^{i\gamma\psi_3}|\phi\rangle \\
{}_-\langle ph|ph\rangle_- &= -{}_+\langle ph|ph\rangle_+
\end{aligned}
\tag{3.13}
$$

where it is natural to restrict the ranges to the group manifold. $0\leq\beta\leq\pi$, $-\pi\leq\alpha,\gamma\leq\pi$ (SO(3)) or $-2\pi\leq\alpha\pm\gamma\leq2\pi$ (SU(2)) which corresponds to $0\leq u\leq\pi$ in (3.8).

4 SL(2,R) gauge theories

In this section we consider some general properties of an $SL(2,R)$ gauge theory. We consider the case when the gauge generators ψ_a satisfy the algebra

$$[\psi_a, \psi_b]_- = i\varepsilon_{ab}{}^c\psi_c \tag{4.1}$$

where we make use of the pseudo-euclidean metric η_{ab} (diag $\eta_{ab} = (-1, 1, 1)$). The BRST charge operator is

$$Q = \psi_a\eta^a - \frac{1}{2}i\varepsilon_{bc}{}^a\mathcal{P}_a\eta^b\eta^c + \bar{\mathcal{P}}_a\pi^a \tag{4.2}$$

With the gauge fixing $\rho = \mathcal{P}_a v^a$ we find

$$[\rho, Q]_+ = \psi_a v^a + \frac{1}{2}i\varepsilon_{ab}{}^c(\mathcal{P}_c\eta^b - \eta^b\mathcal{P}_c)v^a + i\mathcal{P}_a\bar{\mathcal{P}}^a \tag{4.3}$$

and from the general formulas (2.14),(2.21),(2.22), and (2.23) we get

$$\begin{aligned} {}_+\langle ph|ph\rangle_+ &= \langle\Phi|e^{[\rho,Q]_+}|\Phi\rangle = \langle\phi|\,{}_\pi\langle 0|e^{\psi_a v^a}\det{(L^{-1})^a}_b(iv)|0\rangle_\pi|\phi\rangle \\ {}_-\langle ph|ph\rangle_- &= \langle\Phi|e^{-[\rho,Q]_+}|\Phi\rangle = \langle\phi|\,{}_\pi\langle 0|e^{-\psi_a v^a}\det{(L^{-1})^a}_b(iv)|0\rangle_\pi|\phi\rangle \end{aligned} \tag{4.4}$$

For $SL(2,R)$ the left invariant vielbein ${(L^{-1})^a}_b(iv)$ has the explicit form
$(v = \sqrt{-(v^1)^2 + (v^2)^2 + (v^3)^2})$

$$(L^{-1})^a{}_b(iv) = \delta^a_b\frac{\sinh v}{v} + \frac{v^a v_b}{v^2}(1 - \frac{\sin v}{v}) - i\varepsilon^a{}_{bc}v^c\frac{\cos v - 1}{v^2} \tag{4.5}$$

from which we find the measure operator

$$\det{(L^{-1})^a}_b(iv) = \frac{2(1-\cos v)}{v^2} \tag{4.6}$$

which satisfies the properties (2.30) and (2.32). Even in this case the eigenvalue of the measure operator is real if v^a for each value of a has either real or imaginary eigenvalues. For real spectra of the Lagrange multipliers v^a we find $(v \equiv \sqrt{-(v^1)^2 + (v^2)^2 + (v^3)^2}\ )$

$$\begin{aligned} {}_+\langle ph|ph\rangle_+ &= \int d^3v\frac{2(1-\cos v)}{v^2}\langle\phi|e^{\psi_a v^a}|\phi\rangle \\ {}_-\langle ph|ph\rangle_- &= -\int d^3v\frac{2(1-\cos v)}{v^2}\langle\phi|e^{-\psi_a v^a}|\phi\rangle \end{aligned} \tag{4.7}$$

and for imaginary spectra ($v^a \rightarrow iu^a$, $u = \sqrt{-(u^1)^2 + (u^2)^2 + (u^3)^2}$)

$$\begin{aligned} {}_+\langle ph|ph\rangle_+ &= \int d^3u\frac{2(\cosh u - 1)}{u^2}\langle\phi|e^{i\psi_a u^a}|\phi\rangle \\ {}_-\langle ph|ph\rangle_- &= -{}_+\langle ph|ph\rangle_+ \end{aligned} \tag{4.8}$$

Now as in the SU(2) case we require the eigenvalues of $v = \sqrt{-(v^1)^2 + (v^2)^2 + (v^3)^2}$ to be either real or imaginary since the transition $v \rightarrow \pm\, iu$ is double valued. This is accomplished in (4.7) by restrictions of the type $v^2 \pm v^1 \geq 0$. It is also obtained if the quantization is chosen so that v^1 has imaginary eigenvalues ($v^1 = iu^1$) and v^2, v^3 real ones ($v = \sqrt{(u^1)^2 + (v^2)^2 + (v^3)^2}$). In this case it is also natural to restrict the ranges

further so that v lies in the range $0 \leq v < 2\pi$ since the measure is zero for $u = 2\pi n$ $(n = \pm 1, \pm 2, ...)$. However, there are no allowed restrictions on u^a which make u real in (4.8) since the ranges of u^a must be symmetric around zero. Thus, in distinction to the SU(2) case one may not choose the natural unitary quantization (4.8) in an SL(2,R) gauge theory! However, a real u is obtained for the following choices of the quantization:

1. v^1 has real eigenvalues and v^2, v^3 imaginary ones
   $(u = \sqrt{(v^1)^2 + (u^2)^2 + (u^3)^2})$
2. v^1, v^2 have real eigenvalues and v^3 imaginary ones together with the restriction $v^1 \pm v^2 \geq 0$
3. v^1, v^3 have real eigenvalues and v^2 imaginary ones together with the restriction $v^1 \pm v^3 \geq 0$

together with combinations of cases 2) and 3). There are no further restrictions in these cases.

We may now proceed and analyze the above cases further by a factorization of the exponential $e^{\psi_a v_a}$ in (4.4). For the case when v^a have real eigenvalues and when v is real it is natural to consider the factorization (case II in Appendix B)

$$e^{\psi_a v_a} = e^{\alpha\psi_3} e^{\beta\psi_2} e^{\gamma\psi_3} \tag{4.9}$$

Real v^a requires real β and $\alpha = \xi + i\zeta$, $\gamma = \xi - i\zeta$ where ξ and ζ are real with the ranges $-\pi \leq \xi \leq \pi$, $-\infty < \zeta < \infty$. The restriction $v^2 \pm v^1 \geq 0$ requires $0 \leq \beta \leq \pi$. Eq.(4.7) reduces to

$$_+\langle ph|ph\rangle_+ = \int d\xi d\zeta d\beta \sin\beta \langle\phi|e^{(\xi+i\zeta)\psi_3} e^{\beta\psi_2} e^{(\xi-i\zeta)\psi_3}|\phi\rangle \tag{4.10}$$

In the case when v^1 has imaginary eigenvalues and v^2, v^3 real ones, one may also make use of the factorization (4.9) but now with α, β, γ real and with the ranges $-\pi \leq \alpha \pm \gamma \leq \pi$, $0 \leq \beta \leq \pi$ which actually requires the additional restriction $v^2 > 0$. Eq.(4.7) reduces then to

$$_+\langle ph|ph\rangle_+ = \int d\alpha d\beta d\gamma \sin\beta \langle\phi|e^{\alpha\psi_3} e^{\beta\psi_2} e^{\gamma\psi_3}|\phi\rangle \tag{4.11}$$

Alternatively one may make use of the factorization (case III in Appendix B)

$$e^{\psi_a v_a} = e^{\alpha\psi_1} e^{\beta\psi_3} e^{\gamma\psi_1} \tag{4.12}$$

which requires $v^3 > 0$. Here the eigenvalues of α and γ must be imaginary and eq.(4.7) reduces to $(\alpha, \gamma \rightarrow i\alpha, i\gamma)$

$$_+\langle ph|ph\rangle_+ = \int d\alpha d\beta d\gamma \sin\beta \langle\phi|e^{i\alpha\psi_1} e^{\beta\psi_3} e^{i\gamma\psi_1}|\phi\rangle \tag{4.13}$$

with the same ranges as in (4.11).

In the case when v^1 is real and v^2, v^3 imaginary and u real (case 1 above) we may use the factorization (case I in Appendix B)

$$e^{\psi_a v_a} = e^{\alpha\psi_3} e^{\beta\psi_1} e^{\gamma\psi_3} \tag{4.14}$$

Here β is real and should be positive or negative. For $\beta > 0$ we must impose the restriction $v^1 > 0$. For $\alpha = \xi + i\zeta$ and $\gamma = -\xi + i\zeta$ with ξ, ζ real the ranges are $-\pi/2 < \xi < \pi/2$, $-\infty < \zeta < \infty$. Eq.(4.8) reduces to

$$_+\langle ph|ph\rangle_+ = \int d\xi d\zeta d\beta \sinh\beta \langle\phi|e^{(\xi+i\zeta)\psi_3}e^{\beta\psi_1}e^{(-\xi+i\zeta)\psi_3}|\phi\rangle \tag{4.15}$$

For the case when v^1, v^2 are real and v^3 imaginary with u real and $v^1 \pm v^2 > 0$ or $v^1 \pm v^2 < 0$ (case 2 above) there are several possible factorizations. We may use (4.14) with β real and α and γ imaginary. The ranges of α and γ are then infinite while $\beta > 0$ for $v^1 \pm v^2 > 0$. Eq.(4.8) reduces to $(\alpha, \gamma \rightarrow i\alpha, i\gamma)$

$$_+\langle ph|ph\rangle_+ = \int d\alpha d\beta d\gamma \sinh\beta \langle\phi|e^{i\alpha\psi_3}e^{\beta\psi_1}e^{i\gamma\psi_3}|\phi\rangle \tag{4.16}$$

In this case we may also make use of the factorizations IV and V in Appendix B, *i.e.*

$$e^{\psi_a v_a} = e^{\alpha\phi_1}e^{\beta\phi_2}e^{\gamma\phi_1}, \quad e^{\psi_a v_a} = e^{\alpha\phi_2}e^{\beta\phi_1}e^{\gamma\phi_2} \tag{4.17}$$

where

$$\phi_1 = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2), \quad \phi_2 = \frac{1}{\sqrt{2}}(\psi_1 - \psi_2) \tag{4.18}$$

α, β and γ are real here and are *e.g.* positive for $v^1 \pm v^2 > 0$. Eq. (4.8) reduces to

$$_+\langle ph|ph\rangle_+ = \int d\alpha d\beta d\gamma \beta \langle\phi|e^{\alpha\phi_1}e^{\beta\phi_2}e^{\gamma\phi_1}|\phi\rangle \tag{4.19}$$

$$_+\langle ph|ph\rangle_+ = \int d\alpha d\beta d\gamma \beta \langle\phi|e^{\alpha\phi_2}e^{\beta\phi_1}e^{\gamma\phi_2}|\phi\rangle \tag{4.20}$$

At this general level one cannot proceed further except to discuss more factorizations. Now it depends on the properties of the specific model whether or not any of the above cases lead to finite (positive) physical norms.

5 A specific SL(2,R) model

Consider a finite dimensional model with dimension n spanned by the coordinates y^A and momenta p_A $(A = 1, ..., n)$. Let the indices be raised and lowered by the constant symmetric metric matrices η^{AB} and η_{AB} respectively ($\eta^{AB}\eta_{BC} = \delta^A_C$). On this phase space one may naturally introduce an SL(2,R) gauge symmetry which is invariant under the global symmetry implied by the metric η^{AB} through the following constraints

$$p^2 \equiv p_A p_B \eta^{AB} = 0, \quad y^2 \equiv y^A y^B \eta_{AB} = 0, \quad p \cdot y \equiv p_A y^A = 0 \tag{5.1}$$

Such a model is described by the phase space Lagrangian

$$L = p_A \dot{y}^A - \lambda_1 p^2 - \lambda_2 y^2 - \lambda_3 p \cdot y \tag{5.2}$$

where λ_a are Lagrange multipliers. A configuration space Lagrangian is obtained by an elimination of p_A which is possible if $\lambda_1 \neq 0$ which we assume. In [10, 11] such a model

with an O(4,2) metric was proposed and analyzed. It describes both a massless and a massive free relativistic particle depending on the choice of gauge.

Imposing the canonical commutation relations

$$[y^A, p_B]_- = i\delta^A_B \tag{5.3}$$

one easily finds that the normalized constraint operators

$$\phi_1 = \frac{1}{2\sqrt{2}}p^2, \quad \phi_2 = \frac{1}{2\sqrt{2}}y^2, \quad \phi_3 = \frac{1}{4}(p\cdot y + y\cdot p) \tag{5.4}$$

satisfy the commutator algebra

$$[\phi_1, \phi_2]_- = -i\phi_3, \quad [\phi_2, \phi_3]_- = i\phi_2, \quad [\phi_3, \phi_1]_- = i\phi_1 \tag{5.5}$$

This is an SL(2,R) algebra. To see this one may consider the following redefined constraint operators

$$\begin{aligned}
&\psi_1 = \frac{1}{\sqrt{2}}(\phi_1 + \phi_2) = \frac{1}{4}(p^2 + y^2), \quad \psi_2 = \frac{1}{\sqrt{2}}(\phi_1 - \phi_2) = \frac{1}{4}(p^2 - y^2),\\
&\psi_3 = \phi_3 = \frac{1}{4}(p\cdot y + y\cdot p)
\end{aligned} \tag{5.6}$$

ψ_a satisy then the standard SL(2,R) algebra (4.1).

Now one may notice that ϕ_i in (5.4) have exactly the same relation to ψ_a as $\phi_{1,2}$ in (4.18). Hence, the factorizations (4.17) are very natural choices here. The expressions (4.19) and (4.20) may be integrated (for $n \geq 5$) after insertion of the completeness relations

$$\int d^n y |y\rangle\langle y| = \mathbf{1}, \quad \int d^n p |p\rangle\langle p| = \mathbf{1} \tag{5.7}$$

for real eigenvalues and with the modifications implied by (2.28) for imaginary eigenvalues. We find

$$\begin{aligned}
{}_\pm\langle ph|ph\rangle_\pm &= \int d^n y d^n y' \frac{\phi(y)^*}{y^2}\frac{1}{(y-y')^{2(\frac{n}{2}-2)}}\frac{\phi(y')}{y'^2}\\
{}_\pm\langle ph|ph\rangle_\pm &= \int d^n p d^n p' \frac{\phi(p)^*}{p^2}\frac{1}{(p-p')^{2(\frac{n}{2}-2)}}\frac{\phi(p')}{p'^2}
\end{aligned} \tag{5.8}$$

for appropriate choices of the signs of α, β and γ (which is opposite for $|ph\rangle_+$ and $|ph\rangle_-$) and provided $y^2 > 0$, $y'^2 > 0$, $(y-y')^2 > 0$ and $p^2 > 0$, $p'^2 > 0$, $(p-p')^2 > 0$ respectively. Since two of the Lagrange multipliers have real eigenvalues, the quantization rule (2.33) requires two of the coordinate and momentum components to be quantized with imaginary eigenvalues. This together with the requirement $y^2 > 0$ and $p^2 > 0$ etc imply that (5.8) is only valid for models with a global $O(n-2, 2)$ invariance where $n \geq 5$. Also the factorization (4.14) is possible for the above cases and should yield the result (5.8) as well. However, (4.16) is more complicated than (4.19)-(4.20) and we have not worked it out explicitly. We notice though that the β integration is only finite for $p^2 + y^2 > 0$ and for dimensions $n \geq 5$ (the vacuum energy for $p^2 + y^2$ is $n/2$ and yields $e^{\beta\frac{n}{4}}$ which together with the factor $e^{-\beta}$ from $\sinh\beta$ must lead to $e^{\beta\delta}$, $\delta > 0$).

The physical norms (5.8) are actually positive. This is easily seen by means of a Fourier transformation. We find

$$
\begin{aligned}
{}_{\pm}\langle ph|ph\rangle_{\pm} &= \int d^n p \tilde{\phi}(p)^* \frac{1}{(p^2)^2} \tilde{\phi}(p) \\
{}_{\pm}\langle ph|ph\rangle_{\pm} &= \int d^n y \tilde{\phi}(y)^* \frac{1}{(y^2)^2} \tilde{\phi}(y)
\end{aligned}
\tag{5.9}
$$

where $\tilde{\phi}(p)$ and $\tilde{\phi}(y)$ are Fourier transforms of $\phi(y)/y^2$ and $\phi(p)/p^2$ respectively.

In six dimensions ($n = 6$) (5.8) reduces to

$$
\begin{aligned}
{}_{\pm}\langle ph|ph\rangle_{\pm} &= \int d^6 y d^6 y' \frac{\phi(y)^*}{y^2} \frac{1}{(y-y')^2} \frac{\phi(y')}{y'^2} \\
{}_{\pm}\langle ph|ph\rangle_{\pm} &= \int d^6 p d^6 p' \frac{\phi(p)^*}{p^2} \frac{1}{(p-p')^2} \frac{\phi(p')}{p'^2}
\end{aligned}
\tag{5.10}
$$

which from the above argument is valid for a model with a global O(4,2) invariance *i.e.* exactly the invariance of the model in [10, 11]. In this model we may look more precisely into the connection between the quantization of Lagrange multipliers and matter coordinates as prescribed by the quantization rule (2.33). In [10, 11] the components A were chosen to run over the values $A = 0, 1, 2, 3, 5, 6$ where y^0 is (related) to the time component. In the reduction to a massive or massless free relativistic particle $p^2,\ p \cdot y,\ y^2$ eliminate the coordinates $y^0,\ y^5$ and y^6 respectively. Since $v^1,\ v^2$ are real and v^3 imaginary we should according to the rule (2.33) choose $y^0,\ y^6$ imaginary which is exactly what is required to have $y^2 > 0$ and $p^2 > 0$ in (5.10) since $\eta_{00} = \eta_{66} = -1$. Since time is imaginary (5.10) represents a euclidean expression for the free relativistic particle both in the massive and in the massless case.

6 An example of a nonunimodular gauge group; the scaling group

A simple gauge group which is not unimodular is generated by ψ_1 and ψ_2 satisfying the Lie algebra

$$
[\psi_1, \psi_2] = i\psi_2 \tag{6.1}
$$

Since ψ_1 generates scaling of ψ_2 we call this group the scaling group. The BRST charge operator (2.2) is in this case

$$
Q = \psi_a \eta^a - i\mathcal{P}_2 \eta^1 \eta^2 - \frac{i}{2}\eta^1 + \pi_a \bar{\mathcal{P}}^a \tag{6.2}
$$

The gauge fixing operator $\rho = \eta^a \mathcal{P}_a$ leads then to the expressions (2.14) and (2.17) where

$$
\begin{aligned}
&(L^{-1})^a{}_b(iv) = \delta^a_1 \delta^1_b + \delta^a_2 \delta^2_b \frac{i}{v^1}(1 - e^{iv^1}) + \\
&+\delta^a_1 \delta^2_b \frac{v^2}{v^1}(\frac{i}{v^1}(e^{iv^1} - 1) + 1)
\end{aligned}
\tag{6.3}
$$

The scaling group is not unimodular since

$$\det{(L^{\dagger -1})^a}_b \neq \det{(L^{-1})^a}_b, \quad {U_{1a}}^a = 1 \tag{6.4}$$

which violates (2.30). As a consequence we obtain the two formally different expressions in (2.22) and (2.23) for ${}_{\pm}\langle ph|ph\rangle_{\pm}$. However, inserting the explicit expression (6.3) we find the unique answer

$${}_{\pm}\langle ph|ph\rangle_{\pm} = \langle\phi|\,{}_{\pi}\langle 0|e^{\pm\psi_a v^a}\frac{2}{v^1}\sin\frac{v^1}{2}|0\rangle_{\pi}|\phi\rangle \tag{6.5}$$

Thus, the difference in $\det{(L^{\dagger -1})^a}_b$ and $\det{(L^{-1})^a}_b$ is compensated by the factors caused by ${U_{ab}}^b \neq 0$. For real spectra of v^a we have in particular

$${}_{\pm}\langle ph|ph\rangle_{\pm} = \int dv^1 dv^2 \frac{2}{v^1}\sin\frac{v^1}{2}\langle\phi|e^{\pm\psi_a v^a}|\phi\rangle \tag{6.6}$$

and for imaginary spectra ($v^a = iu^a$)

$${}_{\pm}\langle ph|ph\rangle_{\pm} = \int du^1 du^2 \frac{2}{u^1}\sinh\frac{u^1}{2}\langle\phi|e^{\pm i\psi_a u^a}|\phi\rangle \tag{6.7}$$

We may also have a mixture, *i.e.* v^1 real and v^2 imaginary or vice versa. For v^1 real the measure is zero for $v^1 = 2\pi n$ where n is a nonzero integer. Hence, in this case it is natural to restrict v^1 to *e.g.* $-2\pi < v^1 < 2\pi$.

To proceed and to make still simpler expressions we need to factorize $e^{\pm\psi_a v^a}$. It is straight-forward to derive

$$e^{\pm\psi_a v^a} = e^{\pm\frac{v^1}{2}\psi_1}e^{\pm\beta\psi_2}e^{\pm\frac{v^1}{2}\psi_1} \tag{6.8}$$

where

$$\beta = \left(\frac{2}{v^1}\sin\frac{v^1}{2}\right)v^2 \tag{6.9}$$

which is real or imaginary for v^2 real or imaginary. Hence, depending on the spectra of the Lagrange multipliers v^a we arrive at the following formulas for ${}_{\pm}\langle ph|ph\rangle_{\pm}$: For real v^a

$${}_{\pm}\langle ph|ph\rangle_{\pm} = \int dv^1 d\beta \langle\phi|e^{\pm\frac{v^1}{2}\psi_1}e^{\pm\beta\psi_2}e^{\pm\frac{v^1}{2}\psi_1}|\phi\rangle \tag{6.10}$$

and for imaginary v^a we have ($v^1 \rightarrow iu^1$, $\beta \rightarrow i\beta$)

$${}_{\pm}\langle ph|ph\rangle_{\pm} = \int du^1 d\beta \langle\phi|e^{\pm i\frac{u^1}{2}\psi_1}e^{\pm i\beta\psi_2}e^{\pm i\frac{u^1}{2}\psi_1}|\phi\rangle \tag{6.11}$$

For the mixtures we have the obvious combinations.

7 Explicit realizations of the scaling group

Consider a matter state space spanned by the hermitian coordinate and momentum operators y^A and p_A ($A = 1, \ldots, n$) satisfying the commutation relations (5.3). As in section 5 we let the indices be raised and lowered by the real constant symmetric metric η^{AB}, η_{AB} ($\eta^{AB}\eta_{BC} = \delta^A_C$). On this phase space the hermitian gauge generators ψ_1 and ψ_2 satisfying the Lie algebra (6.1) may be realized by the expressions

$$\psi_1 = \frac{1}{4}(p \cdot y + y \cdot p), \quad \psi_2 = \alpha p^2 \tag{7.1}$$

where α is a real constant different from zero. In [12] a five dimensional model of this kind was proposed. It has an O(4,1) metric and was shown to describe either a free massive relativistic particle or a massless one in a de Sitter space depending on the choice of gauge.

In terms of this explicit realization we may now further simplify the general formulas (6.10)-(6.11). Due to the form of ψ_2 in (7.1) it is natural to write these expressions in terms of wave functions in momentum space by means of the completeness relation

$$\int d^n p |p\rangle\langle p| = \mathbf{1} \tag{7.2}$$

, for real p_A and with the modification implied by (2.28) for imaginary ones. Since

$$\langle p|e^{\pm\frac{v^1}{2}\psi_1}|\phi\rangle = \langle e^{\pm i\frac{v^1}{4}}p|\phi\rangle \tag{7.3}$$

only imaginary spectra of v^1 leads to wave functions with real arguments. For imaginary v^1 and v^2 ($v^a \rightarrow iu^a$) we find from (6.11)

$$\begin{aligned} {}_\pm\langle ph|ph\rangle_\pm &= \int du^1 d\beta d^n p \, \phi^*(e^{\pm\frac{u^1}{4}}p)e^{\pm i\beta\alpha p^2}\phi(e^{\mp\frac{u^1}{4}}p) = \\ &= \frac{4}{|\alpha|}\int d\gamma d^n p \, \phi^*(e^{\pm\gamma}p)\delta(p^2)\phi(e^{\mp\gamma}p) \end{aligned} \tag{7.4}$$

and for real v^2 restricted to positive or negative eigenvalues

$${}_\pm\langle ph|ph\rangle_\pm = \frac{4}{|\alpha|}\int d\gamma \frac{d^n p}{p^2}\phi^*(e^{\pm\gamma}p)\phi(e^{\mp\gamma}p) \tag{7.5}$$

This construction requires $p^2 > 0$ and since only one momentum component is quantized with imaginary eigenvalues according to the general quantization rule, the expression (7.5) is only valid for an $O(n-1,1)$ metric η_{AB} and in particular for the $O(4,1)$ model considered in [12]. Although all momentum components are to be quantized with real eigenvalues in (7.4) only an $O(n-1,1)$ metric makes the scalar product (7.4) nonzero and well defined. Thus, only for an $O(n-1,1)$ metric η_{AB} do we find well defined physical scalar products. We claim that the norms (7.4) and (7.5) are positive when they exist.

8 Conclusions

In this paper we have dealt with both general and specific properties of gauge theories with finite number of degrees of freedom and when quantized on inner product spaces by means

of a BRST procedure. For Lie group theories we derived general geometrical expressions for the norms of the physical states found in [1]. We obtained expressions involving only matter states, gauge generators and vielbeins on the group manifold, (2.26)-(2.27). When the Lagrange multipliers were quantized with indefinite metric states we obtained the most natural geometrical expressions (2.29) which involve the group measures and which we expect to be useful for compact Lie groups like $SU(2)$. However, for noncompact Lie groups like $SL(2,R)$ these natural expressions are inappropriate. In this case some Lagrange multipliers must be quantized with positive metric states. We have performed a detailed analysis of $SL(2,R)$ theories in order to determine the possible quantization choices of the Lagrange multipliers. These formulas were then applied to a simple class of $SL(2,R)$ theories. In this way we found physical states of the BRST quantization for these models which have positive norms. It would be nice if our general analysis of the $SL(2,R)$ case could be endowed with a natural geometrical setting which in turn could serve as a guide to more complex noncompact gauge groups.

Since our general formulas are also valid for gauge groups which are not unimodular we analysed the simple scaling group which involves two generators and applied the resulting formulas to a simple class of models with this invariance. Physical states with positive norms were then obtained.

For the cases in which the general formulas (2.26)-(2.27) are not applicable we expect that the procedure of ref.[6] should work instead. This procedure involves a Dirac quantization. If neither (2.26)-(2.27) nor ref.[6] nor any combination of the two procedures is applicable we believe that a BRST quantization is not possible at all.

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Appendix A

Derivation of the general expressions (2.14) and (2.17) in section 2.

From (2.13) we have

$$e^{[\rho,Q]_+}=e^{\psi'_a v^a}e^{a+b}=e^{\psi''_a v^a}e^{a+c} \tag{A.1}$$

where

$$\begin{aligned}&\psi'_a\equiv\psi_a-\frac{1}{2}iU_{ab}{}^b,\quad \psi''_a\equiv\psi_a+\frac{1}{2}iU_{ab}{}^b=(\psi'_a)^\dagger,\\ &a\equiv -i\mathcal{P}_a\bar{\mathcal{P}}^a,\quad b\equiv iU_{ab}{}^c\mathcal{P}_c\eta^b,\quad c\equiv -iU_{ab}{}^c\eta^b\mathcal{P}_c=b^\dagger\end{aligned} \tag{A.2}$$

Define now

$$F(\lambda)\equiv e^{\lambda(a+b)}e^{-\lambda b},\quad G(\lambda)\equiv e^{-\lambda c}e^{\lambda(a+c)} \tag{A.3}$$

Then we have

$$F(1)|\Phi\rangle=e^{a+b}|\Phi\rangle,\quad \langle\Phi|G(1)=\langle\Phi|e^{a+c} \tag{A.4}$$

due to the property (2.9). $F(\lambda)$ and $G(\lambda)$ are easily calculated. We have *e.g.*

$$F'(\lambda)=F(\lambda)A(\lambda)=B(\lambda)F(\lambda) \tag{A.5}$$

where

$$A(\lambda)\equiv e^{\lambda b}ae^{-\lambda b},\quad B(\lambda)\equiv e^{\lambda(a+b)}ae^{-\lambda(a+b)} \tag{A.6}$$

One may easily show that

$$A(\lambda)=B(\lambda)=e^{\lambda[\rho,Q]}ae^{-\lambda[\rho,Q]} \tag{A.7}$$

Hence

$$F(\lambda)=e^{\int_0^\lambda d\lambda' A(\lambda')} \tag{A.8}$$

$A(\lambda)$ is explicitly (see also [1])

$$A(\lambda)=-iA^a{}_b(i\lambda v)\mathcal{P}_a\bar{\mathcal{P}}^b \tag{A.9}$$

where $A^a{}_b(i\lambda v^a)$ is the adjoint matrix representation of the gauge group with the imaginary group coordinates $i\lambda v^a$. Now we have the formula

$$(L^{-1})^a{}_b(iv)=\int_0^1 d\lambda A^a{}_b(i\lambda v) \tag{A.10}$$

where $L^a{}_b(iv)$ is the left invariant vielbein on the group manifold with imaginary coordinates iv^a. Hence

$$F(1)=e^{-i(L^{-1})^a{}_b(iv)\mathcal{P}_a\bar{\mathcal{P}}^b} \tag{A.11}$$

Similarly one may show that

$$G(1)=e^{-i(L^{\dagger-1})^a{}_b(iv)\mathcal{P}_a\bar{\mathcal{P}}^b}=(F(1))^\dagger \tag{A.12}$$

where

$$L^{\dagger a}{}_b(iv) = L^a{}_b(-iv) \tag{A.13}$$

is the right invariant vielbein on the group manifold. Formulas (2.14) follow now.

For $e^{-[\rho,Q]_+}$ we have

$$e^{-[\rho,Q]_+} = e^{-\psi'_a v^a} e^{-a-b} = e^{-\psi''_a v^a} e^{-a-c} \tag{A.14}$$

which implies

$$F(-1)|\Phi\rangle = e^{-a-b}|\Phi\rangle, \quad \langle\Phi|G(-1) = \langle\Phi|e^{-a-c} \tag{A.15}$$

where

$$\begin{aligned} G(-1) &= e^{-\int_0^1 d\lambda A(\lambda)} = e^{i(L^{-1})^a{}_b(iv)\mathcal{P}_a\bar{\mathcal{P}}^b}, \\ F(-1) &= e^{i(L^{\dagger -1})^a{}_b(iv)\mathcal{P}_a\bar{\mathcal{P}}^b} = (G(-1))^\dagger \end{aligned} \tag{A.16}$$

Formulas (2.17) follow now.

Appendix B

Some factorizations for SL(2,R).

Let ψ_a be the gauge generators for SL(2,R) satisfying the algebra (4.1). The following factorizations of $e^{\psi_a v_a}$ in (4.4) are considered in section 4:
Case I:

$$e^{\psi_a v_a} = e^{\alpha\psi_3} e^{\beta\psi_1} e^{\gamma\psi_3} \tag{B.1}$$

where α, β and γ are expressions in terms of v_a. The right-hand side is manifestly hermitian if

$$\beta^\dagger = \beta, \quad \gamma = \alpha^\dagger \tag{B.2}$$

which we require. Eq.(B.1) is easily solved algebraically. We find

$$\frac{v^1}{v}\sin\frac{v}{2} = \sinh\frac{\beta}{2}\cos\frac{\alpha-\gamma}{2} \tag{B.3}$$

$$\frac{v^2}{v}\sin\frac{v}{2} = i\sinh\frac{\beta}{2}\sin\frac{\alpha-\gamma}{2} \tag{B.4}$$

$$\frac{v^3}{v}\sin\frac{v}{2} = \cosh\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2} \tag{B.5}$$

$$\cos\frac{v}{2} = \cosh\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}. \tag{B.6}$$

All these relations are consistent with hermitian v^a and (B.2). For the measure we have formally

$$d^3v\frac{2(1-\cos v)}{v^2} = id\alpha d\beta d\gamma \sinh\beta \tag{B.7}$$

Case II:

$$e^{\psi_a v_a} = e^{\alpha\psi_3}e^{\beta\psi_2}e^{\gamma\psi_3} \tag{B.8}$$

which requires (B.2) and

$$\frac{v^1}{v}\sin\frac{v}{2} = i\sin\frac{\beta}{2}\sin\frac{\alpha-\gamma}{2} \tag{B.9}$$

$$\frac{v^2}{v}\sin\frac{v}{2} = \sin\frac{\beta}{2}\cos\frac{\alpha-\gamma}{2} \tag{B.10}$$

$$\frac{v^3}{v}\sin\frac{v}{2} = \cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2} \tag{B.11}$$

$$\cos\frac{v}{2} = \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}. \tag{B.12}$$

For the measure we find the formal relation

$$d^3v\frac{2(1-\cos v)}{v^2} = -id\alpha d\beta d\gamma \sin\beta \tag{B.13}$$

Case III:

$$e^{\psi_a v_a} = e^{\alpha\psi_1}e^{\beta\psi_3}e^{\gamma\psi_1} \tag{B.14}$$

which requires (B.2) and

$$\frac{v^1}{v}\sin\frac{v}{2} = \cos\frac{\beta}{2}\sinh\frac{\alpha+\gamma}{2} \tag{B.15}$$

$$\frac{v^2}{v}\sin\frac{v}{2} = -i\sin\frac{\beta}{2}\sinh\frac{\alpha-\gamma}{2} \tag{B.16}$$

$$\frac{v^3}{v}\sin\frac{v}{2} = \sin\frac{\beta}{2}\cosh\frac{\alpha-\gamma}{2} \tag{B.17}$$

$$\cos\frac{v}{2} = \cos\frac{\beta}{2}\cosh\frac{\alpha+\gamma}{2}. \tag{B.18}$$

For the measure we find

$$d^3v\frac{2(1-\cos v)}{v^2} = id\alpha d\beta d\gamma \sin\beta \tag{B.19}$$

Case IV:

$$e^{\psi_a v_a} = e^{\alpha\phi_1}e^{\beta\phi_2}e^{\gamma\phi_1} \tag{B.20}$$

where

$$\phi_1 = \frac{1}{\sqrt{2}}(\psi_1+\psi_2), \quad \phi_2 = \frac{1}{\sqrt{2}}(\psi_1-\psi_2), \quad \phi_3 = \psi_3 \tag{B.21}$$

Eq.(B.20) requires (B.2) and

$$\frac{v^1+v^2}{v}\sin\frac{v}{2} = \frac{\sqrt{2}}{2}(\alpha+\gamma+\frac{1}{2}\alpha\beta\gamma) \tag{B.22}$$

$$\frac{v^1-v^2}{v}\sin\frac{v}{2} = \frac{\sqrt{2}}{2}\beta, \tag{B.23}$$

$$i\frac{v^3}{v}\sin\frac{v}{2} = \frac{1}{4}\beta(\alpha-\gamma), \tag{B.24}$$

$$\cos\frac{v}{2} = \frac{1}{4}(4+\beta(\alpha+\gamma)) \tag{B.25}$$

For the measure we find

$$d^3v\frac{2(1-\cos v)}{v^2} = -id\alpha d\beta d\gamma\beta \tag{B.26}$$

Case V:

$$e^{\psi_a v_a} = e^{\alpha\phi_2}e^{\beta\phi_1}e^{\gamma\phi_2} \tag{B.27}$$

which requires (B.2) and

$$\frac{v^1+v^2}{v}\sin\frac{v}{2} = \frac{\sqrt{2}}{2}\beta \tag{B.28}$$

$$\frac{v^1-v^2}{v}\sin\frac{v}{2} = \frac{\sqrt{2}}{2}(\alpha+\gamma+\frac{1}{2}\alpha\beta\gamma), \tag{B.29}$$

$$i\frac{v^3}{v}\sin\frac{v}{2} = -\frac{1}{4}\beta(\alpha-\gamma), \tag{B.30}$$

and (B.25). For the measure we find the relation (B.26) here as well.

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